Stability of Dynamic Systems with Periodically Varying Parameters

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A method is developed for analyzing the stability of dynamic systems, such as spinning satellites in elliptic orbits and helicopter rotors in forward flight, which are represented by a coupled set of linear differential equations with periodically varying coefficients. The problem is formulated in terms of a Hill-type infinite determinant, which must vanish if a nontrivial solution is to exist. It is shown that the infinite determinant is equal to a finite sum of known hyperbolic functions. Results of calculations of the aeroelastic stability of a helicopter rotor in forward flight are in qualitative agreement with experimental and numerically derived data.

Introduction

A NUMBER of dynamic systems, the stability of which must be considered in their design, have parameters which are periodic functions of time. A blade of a helicopter rotor in forward flight experiences a relative freestream which varies periodically, making the factor of proportionality of the aerodynamic forces a periodic parameter.[1] A spinning satellite in an elliptic orbit passes through a periodically varying gravitational field, with the result that the gravitational forces affecting the attitude of the satellite are periodic functions of time.[2] Rotating machinery with shafts in flexible bearings, the stiffnesses of which are different in two different directions, experiences a periodically varying resistance to its motions.[3] This study was directed to developing a method for analyzing the stability of dynamic systems with periodic parameters.

For purposes of examining stability, these systems are represented by linear differential equations with periodically varying coefficients. Differential equations of this type have been the concern of applied mathematicians for over a century. The differential equation of second order bearing his name was discussed by Mathieu in 1868 in reference to the problem of a vibrating elliptic membrane. The more general second-order equation derived by Hill for determining the motion of the lunar perigee was presented by him in 1877. In 1883, Floquet determined the form of the solution for any linear differential equation with periodic coefficients.†

More recently, Hsu[6] treated a general Nth-order dynamic system with periodic parameters. Modi and Neilson[7] analyzed a spinning satellite in an elliptic orbit, and Coleman and Feingold[8] studied mechanical instabilities of helicopter rotors supported by anisotropic bearings. Aeroelastic stability of a helicopter rotor in forward flight was investigated by Jenkins[9] and by Gates and DuWaldt.[1] These studies and similar ones treating related problems generally rely on expansion in a small parameter to obtain a solution,[6,7] are restricted to special forms of the differential equations,[8] or only generate the solution for a specific configuration, with given initial conditions, by a direct integration in time on an analog or digital computer.[1,9]

The analysis which follows is directed to determining the stability of a dynamic system that is represented by a set of linear differential equations with periodic coefficients. The system is required to have a finite number of degrees of freedom and certain requirements concerning continuity are placed on the coefficients of the equations of motion. The problem, however, is not otherwise restricted.

Preliminary Formulations

The equations of motion as derived, say, from Lagrange's equations, for a linear system with periodically varying parameters and having N degrees of freedom, can be written in the form

$$\frac{d^2\zeta_m}{dz^2} + \sum_{n=1}^{N}\left[a_{mn}\frac{d\zeta_n}{dz} + b_{mn}\zeta_n\right] = 0 \qquad m = 1,2,\ldots,N \tag{1}$$

where a_{mn} and b_{mn} are periodic functions:

$$a_{mn}(z+\pi) = a_{mn}(z),\; b_{mn}(z+\pi) = b_{mn}(z)$$

There is no loss of generality by omitting mass coupling, since those terms can always be eliminated by suitably redefining the dependent variables.

In order to secure the theoretical basis for the solution, it is necessary to first operate on Eqs. (1) to obtain two related sets of equations. It is convenient for this purpose to use matrix notation. If X denotes a column matrix with the dependent variables as elements, and if A and B are $N \times N$ square matrices with elements a_{mn} and b_{mn}, respectively, then Eq. (1) can be written

$$d^2X/dz^2 + AdX/dz + BX = 0 \tag{2}$$

where the derivative of a matrix is the matrix formed by replacing each element by its derivative.

The first step in obtaining the two related sets of equations is to differentiate Eq. (2), yielding

$$d^3X/dz^3 + Ad^2X/dz^2 + [dA/dz + B]dX/dz + [dB/dz]X = 0 \tag{3}$$

If Eq. (2) is solved for d^2X/dz^2 and the result substituted in Eq. (3), it is found that

$$d^3X/dz^3 + CdX/dz + DX = 0 \tag{4}$$

Presented as Paper at the AIAA Structural Dynamics and Aeroelasticity Specialist Conference, New Orleans, La., April 16–17, 1969 (no paper number; published in bound volume of conference papers); submitted April 29, 1969; revision received February 3, 1970. This study was supported by NASA Langley Research Center, under Contract NAS1-7411, while the author was employed by Rochester Applied Science Associates, Rochester, N.Y.

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† A discussion of the early history of Mathieu and related functions is given by McLachlan.[4] Many papers of historical interest are also noted by Whittaker and Watson.[5]

where

$$C = dA/dz + B - A^2, \quad D = dB/dz - AB$$

If, now, Eq. (4) is differentiated and the second derivative of X eliminated as before, it is found that

$$d^4X/dz^4 + EdX/dz + FX = 0 \tag{5}$$

where

$$E = dC/dz + D - CA, \quad F = dD/dz - CB$$

It can be shown[10] that a set of functions is a solution of the original equation [Eq. (2)], if and only if it is a solution to both of the derived equations, Eq. (4) and Eq. (5). As a result, the solutions of Eq. (2) can be found by solving Eqs. (4) and (5) and identifying those solutions common to the latter two equations. That procedure will be followed here, for a reason which will be evident subsequently.

Development of the Infinite Determinant

Consider, first, the solution to Eq. (4). It is convenient at this point to abandon the matrix notation. Thus, if c_{mn} and d_{mn} denote the elements of matrices C and D, respectively, Eq. (4) gives that

$$\frac{d^3\zeta_m}{dz^3} + \sum_{n=1}^{N}\left[c_{mn}\frac{d\zeta_n}{dz} + d_{mn}\zeta_n\right] = 0, \qquad m = 1,2,\ldots,N \tag{6}$$

From the theory of Floquet,[5] the solution of Eqs. (6) must be of the form

$$\zeta_m = e^{\lambda z}\phi_m(z)$$

where $\phi_m(z)$ is periodic, with a period π, and λ is a complex constant. Since the differential system, Eqs. (6), is of order $3N$, there will be $3N$ values of λ defining the solution, $2N$ of which also define the solution of Eq. (1). If any one of these latter characteristic values has a positive real part, the system being analyzed is unstable.

If ϕ_m is expanded in a complex Fourier series, then ζ_m can be written

$$\zeta_m = \sum_{k=-\infty}^{\infty} p_{mk}e^{(2ik+\lambda)z} \tag{7}$$

Also, assuming that the periodic coefficients in Eqs. (6) are suitably well behaved, they can be similarly expanded:

$$c_{mn} = \sum_{k=-\infty}^{\infty} \xi_{mnk}e^{2ikz}, \quad d_{mn} = \sum_{k=-\infty}^{\infty} \eta_{mnk}\, e^{2ikz}$$

$$m,n = 1,2,\ldots,N$$

If Eq. (7) and the Fourier representations for the coefficients are substituted in Eq. (6) and the coefficients of e^{2ikz} are grouped and set equal to zero, a set of linear recursion relations in the unknown coefficients p_{mk} is obtained. Specifically, it is found that

$$0 = [2in + \lambda]^3 p_{rn} + \sum_{s=1}^{N}\left\{\sum_{k=-\infty}^{\infty}[\eta_{rsk} + (2in - 2ik + \lambda)\xi_{rsk}]p_{s,n-k}\right\} \tag{8}$$

$$n = \ldots,-2,-1,0,1,2,\ldots; \quad r = 1,2,\ldots,N$$

These relations constitute an infinite set of linear algebraic equations in the unknown Fourier coefficients p_{mk}. For this set of equations to have a nontrivial solution, the associated infinite determinant $\Delta(\lambda)$ must vanish (a discussion of infinite determinants is given by Whittaker and Watson[5]). The requirement that Δ vanish is the condition which determines λ, and hence the stability of the system.

$$\Delta(\omega) = \begin{vmatrix} & & \vdots & & \vdots & & \vdots & \\ & 1 & \alpha_{12}(-1,-1) & \alpha_{11}(-1,0) & \alpha_{12}(-1,0) & \alpha_{11}(-1,1) & \alpha_{12}(-1,1) & \\ \cdots & \alpha_{21}(-1,-1) & 1 & \alpha_{21}(-1,0) & \alpha_{22}(-1,0) & \alpha_{21}(-1,1) & \alpha_{22}(-1,1) & \cdots \\ & \alpha_{11}(0,-1) & \alpha_{12}(0,-1) & 1 & \alpha_{12}(0,0) & \alpha_{11}(0,1) & \alpha_{12}(0,1) & \\ \cdots & \alpha_{21}(0,-1) & \alpha_{22}(0,-1) & \alpha_{21}(0,0) & 1 & \alpha_{21}(0,1) & \alpha_{22}(0,1) & \cdots \\ & \alpha_{11}(1,-1) & \alpha_{12}(1,-1) & \alpha_{11}(1,0) & \alpha_{12}(1,0) & 1 & \alpha_{12}(1,1) & \\ \cdots & \alpha_{21}(1,-1) & \alpha_{22}(1,-1) & \alpha_{21}(1,0) & \alpha_{22}(1,0) & \alpha_{21}(1,1) & 1 & \cdots \\ & & \vdots & & \vdots & & \vdots & \end{vmatrix}$$

Fig. 1 The arrangement of the determinant elements for the case $N = 2$.

In order that Δ be meaningfully defined, it is necessary to divide each of Eqs. (8) through by the coefficient of p_{rn}. With the unknowns then appropriately ordered, the diagonal elements of Δ are all unity and the off-diagonal elements all vanish in the limit as a row or column index of the determinant tends to either positive or negative infinity. Specifically, the elements $\sigma_{\mu\nu}$ of $\Delta(\lambda)(\mu,\nu = 0,\pm1,\pm2\ldots)$ are given by

$$\sigma_{Nn+r,Nk+s} = \alpha_{rs}(n,k;\lambda)$$

$$n,k = \ldots,-2,-1,0,1,2,\ldots; \quad r,s = 1,2,\ldots,N$$

where

$$\alpha_{rs}(n,k;\lambda) = \frac{(2ik + \lambda)\xi_{rs,n-k} + \eta_{rs,n-k}}{(2in + \lambda)^3 + (2in + \lambda)\xi_{rro} + \eta_{rro}}$$

for $n \neq k$ and/or $r \neq s$, and $\alpha_{rr}(n,n{:}\lambda) = 1$.

The construction of the determinant can perhaps be best visualized as made up of a collection of subarrays which are $N \times N$ in size. The location of any element within a subarray is determined from indices r and s. The location of each subarray is specified through indices n and k. The general arrangement of the α's in Δ is illustrated in Fig. 1 for the case $N = 2$.

The value of Δ is obtained, for a given λ, by evaluating the finite determinant formed by ranging n and k from, say, $-L$ to L. Successively larger values for L are then taken until the limit becomes apparent.[5]

The expression

$$\Delta(\lambda) = 0 \tag{9}$$

which must be satisfied for there to be a solution, by itself is not a particularly useful relation for determining λ, because of the limiting process involved in evaluating infinite determinants. It will be shown, however, that $\Delta(\lambda)$ in fact constitutes a combined series-product expansion in λ of a finite sum of hyperbolic functions. With $\Delta(\lambda)$ expressed in the latter form, Eq. (9) can be solved explicitly for λ.

Derivation of the Equivalent Analytic Form for Δ

Two properties of the function $\Delta(\lambda)$ must first be established. Specifically, it is asserted that 1) $\Delta(\lambda)$ is an analytic function of λ, except for simple, identifiable poles; 2) $\Delta(\lambda)$ is periodic in λ with a period of $2i$.

That $\Delta(\lambda)$ is analytic can be concluded by noting that Δ is an absolutely convergent determinant—i.e., the product of the diagonal elements converges absolutely (in this case to unity) and the double sum of the off-diagonal elements converges absolutely. Uniform convergence and analyticity can then be established.[5]

Note that if the original equations [Eq. (1)], rather than derived equations [Eq. (6)], had been used to generate Δ, that determinant would not have been absolutely convergent since the double sum of off-diagonal elements generated from the original equations is only conditionally convergent.

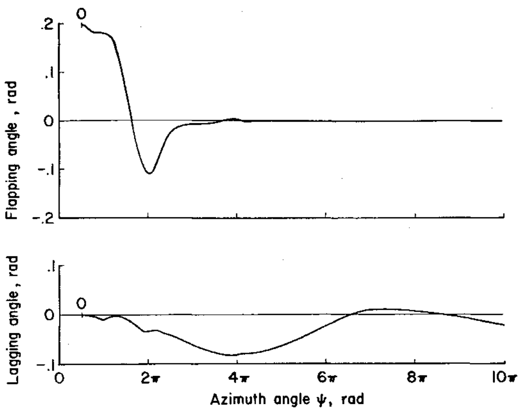

Fig. 2 Blade motions for advance ratio $\mu = 0.6$ (from Jenkins[9]).

What follows hinges on the analyticity of Δ, hence the necessity for working with the differentiated sets of equations.

Clearly, too, the only singularities in Δ are simple poles at the points

$$\lambda = \gamma_{rj} - 2in; \quad n = 0, \pm 1, \pm 2, \ldots; \quad r = 1,2, \ldots N; \quad j = 1,2,3$$

where the γ_{rj}'s are the $3N$ roots of the N cubic equations

$$\gamma^3 + \gamma \xi_{rro} + \eta_{rro} = 0, \quad r = 1,2, \ldots, N$$

The periodicity of Δ is shown by substituting $\lambda + 2i$ for λ in the expression for α_{rs}, whereupon it is found that

$$\alpha_{rs}(n,k;\lambda + 2i) = \alpha_{rs}(n + 1, k + 1; \lambda)$$

Thus, in the limit, the value of Δ is unchanged if λ is replaced by $\lambda + 2i$:

$$\Delta(\lambda + 2i) = \Delta(\lambda)$$

Now, consider the function $D(\lambda)$, defined by

$$D(\lambda) = \Delta(\lambda) + \sum_{r=1}^{N} \sum_{j=1}^{3} f_{rj} \coth\left[\frac{\pi}{2}(\lambda - \gamma_{rj})\right]$$

where the f_{rj}'s are constants. Observe that $D(\lambda)$ is an analytic function and that

$$D(\lambda + 2i) = D(\lambda)$$

Further, note that the term

$$f_{rj} \coth[(\pi/2)(\lambda - \gamma_{rj})]$$

has simple poles at $\lambda = \gamma_{rj} - 2in$ for $n = 0, \pm 1, \pm 2, \ldots$, and has no other singularities. Thus, if the value for each constant f_{rj} is properly chosen, $D(\lambda)$ will have no poles. Let the f_{rj}'s be so chosen, making $D(\lambda)$ analytic throughout the finite part of the λ-plane. But D is clearly bounded at infinity; $\Delta(\lambda)$ has a limit of one and the hyperbolic cotangents have limits of ± 1 for $Re(\lambda) \to \infty$. Therefore, by Liouville's theorem, $D(\lambda)$ is simply some constant, say c:

$$D(\lambda) = c = \Delta(\lambda) + \sum_{r=1}^{N} \sum_{j=1}^{3} f_{rj} \coth\left[\frac{\pi}{2}(\lambda - \gamma_{rj})\right] \quad (10)$$

It only remains to determine the values of the f_{rj}'s and of c. This is facilitated by first considering the limits of $D(\lambda)$ with infinite λ:

$$\lim_{Re(\lambda) \to +\infty} D(\lambda) = c = 1 + \sum_{r=1}^{N} \sum_{j=1}^{3} f_{rj}$$

$$\lim_{Re(\lambda) \to -\infty} D(\lambda) = c = 1 - \sum_{r=1}^{N} \sum_{j=1}^{3} f_{rj}$$

Clearly, then, $c = 1$ and

$$\sum_{r=1}^{N} \sum_{j=1}^{3} f_{rj} = 0$$

Replacing c by unity and solving for Δ in Eq. (10) gives that

$$\Delta(\lambda) = 1 - \sum_{r=1}^{N} \sum_{j=1}^{3} f_{rj} \coth\left[\frac{\pi}{2}(\lambda - \gamma_{rj})\right]$$

Now, let $3N - 1$ arbitrary (but finite) values of λ, say $\lambda_1, \lambda_2, \ldots, \lambda_{3N-1}$, be assigned in the above equation. The resulting $3N - 1$ equations, together with the one obtained for infinite λ, provide sufficient relations to solve for the f_{rj}'s. More specifically, those constants are the solution of

$$\sum_{r=1}^{N} \sum_{j=1}^{3} f_{rj} \coth\left[\frac{\pi}{2}(\lambda_k - \gamma_{rj})\right] = 1 - \Delta(\lambda_k)$$

$$k = 1,2, \ldots, 3N - 1; \quad (11)$$

$$\sum_{r=1}^{N} \sum_{j=1}^{3} f_{rj} = 0$$

With the f_{rj}'s known, the determinantal equation, $\Delta(\lambda) = 0$, can be replaced by

$$1 - \sum_{r=1}^{N} \sum_{j=1}^{3} f_{rj} \coth\left[\frac{\pi}{2}(\lambda - \gamma_{rj})\right] = 0 \quad (12)$$

A polynomial of degree $3N$ in $e^{\lambda\pi}$ can be readily derived from Eq. (12). The $3N$ roots of that polynomial determine the characteristic values for Eq. (4).

In the same manner as is outlined above for Eq. (4), the $4N$ characteristic values for Eq. (5) can be obtained. The $2N$ values common to the two sets are those of the original equations, Eq. (1).

Application of the Theory

A digital computer program was prepared to implement the theory for the case of one, two, or three degrees of freedom. The program was used to analyze a helicopter rotor system in forward flight, having two degrees of freedom, for which solutions by direct time integration were available. Also, preliminary calculations were conducted for a model rotor system having three degrees of freedom, for which experimentally obtained flutter boundaries were available. A discussion of the results of the calculations follows. The details of the computational procedures and the derivation of the rotor equations of motion are given by Crimi.[10]

Comparison with Direct Time Integration

The analysis of stability by direct time integration on a digital computer of a rigid rotor blade with flapping and lead-

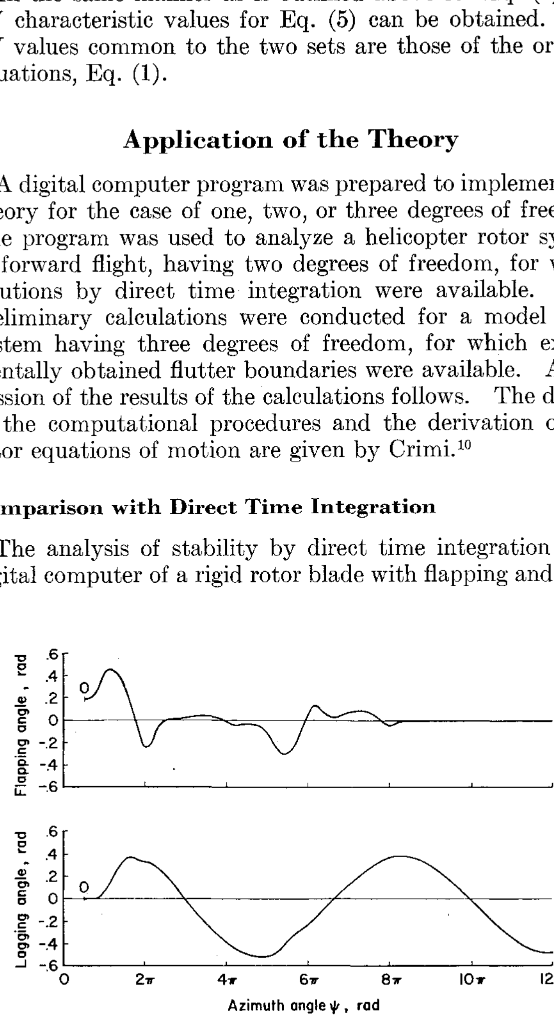

Fig. 3 Blade motions for advance ratio $\mu = 1.4$ (from Jenkins[9]).

lag hinges is reported by Jenkins.[9] The nonlinear representations of both inertial and aerodynamic forces are utilized in the equations of motion. The basic blade for which numerical results were obtained had zero offset of the flapping hinge and $0.05R$ offset of the lead-lag hinge, where R is rotor radius. The blade had a constant chord C except for a cutout from the axis of rotation to $0.2R$. In the calculations reported by Jenkins,[9] the rotor was unloaded and at zero shaft angle. Further details are given in the aforementioned report.

For two of the cases analyzed by Jenkins, the variations of flapping and lead-lag angles with time are presented. These cases had values of advance ratio μ (the ratio $V/\Omega R$, where V is forward speed and Ω is rotor rotational speed) of 0.6 and 1.4, respectively. The mass constant γ' for both cases was 1.6 ($\gamma' = \rho C R^4/I_h$, where I_h is the mass moment of inertia about the lead-lag hinge and ρ is air density). The case for the lower advance ratio is reported to be very stable and the case with $\mu = 1.4$ is indicated to be stable but near a boundary of neutral stability. The time histories of the blade displacements, as given by Jenkins, are reproduced in Figs. 2 and 3.

The appropriate parameter values for these two cases were inserted in linearized equations of motion of a rotor blade developed by Crimi.[10] Fourier series representations of the coefficients in the equations, including the first eleven harmonics of rotor rotational speed, were generated and supplied to the stability-analysis computer program. The characteristic values calculated are given in Table 1, where λ_R and λ_I are the real and imaginary parts, respectively, of λ.

Examination of these results indicates qualitative agreement with the results of the analysis by direct time integration but there is evidence of some differences in quantity. At $\mu = 0.6$, the flapping motion should damp by a factor $e^{-2} = 0.135$ in $4/(-\lambda_R) = 3.527$ rad of azimuth change, by the result obtained here, whereas Fig. 2 indicates a much more rapid decrease. On the other hand, at $\mu = 1.4$, the factor e^{-2} should apply to the flapping motion for a change of $4/(-\lambda_R) = 5.452$ rad, but Fig. 3 indicates that the flapping motion is considerably less stable than that.

From the qualitative viewpoint, the comparison is more favorable. The decrease in stability of the flapping degree of freedom with increasing μ is in evidence in the result obtained here, since λ_R is less negative at $\mu = 1.4$ than at $\mu = 0.6$. The stability of a given system is determined, of course, by the least negative, or most positive, value of λ_R. Also, in agreement with the indications of Figs. 2 and 3, the lead-lag motion is only slightly damped, with a factor e^{-2} decrease occurring in 1130 rad at $\mu = 0.6$ and in 750 rad at $\mu = 1.4$.

The quantitative differences in the predictions of the flapping motion can be attributed to the nonlinear effects which are included in the direct-integration solution, but which are absent from the formulations analyzed here. This can be seen as follows. With the rotor unloaded, as is the case here, the equations for rigid-body flapping and lead-lag motion become decoupled when linearized. Thus, any

Table 1 Characteristic values for comparison with results of direct time integration

μ	Degree of freedom	Characteristic values λ_R	Characteristic values λ_I
0.6	Flapping	−1.134	0.60206
		−1.13400	−0.60206
0.6	Lead-lag	−0.00353	0.56195
		−0.00353	−0.56195
1.4	Flapping	−0.73379	−1.0
		−2.56010	1.0
1.4	Lead-lag	−0.00534	0.56206
		−0.00534	−0.56206

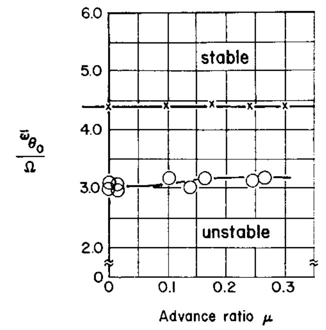

Fig. 4 Experimental flutter boundary for a model rotor blade. Ratio of nonrotating pitch natural frequency $\bar{\omega}_{\theta_0}$ to rotational speed Ω vs advance ratio μ, for $\bar{\omega}_{\phi_1}/\bar{\omega}_{\theta_0} = 1.31$ and $x_e/C = 0.139$ (Gates and DuWaldt[1]). Points at which characteristic values were calculated are indicated by asterisks.

coupling of the motion which is detected can be attributed, in this case, to nonlinear dynamic-coupling effects. Since the motions plotted in Figs. 2 and 3 were initiated by a disturbance in flapping, the considerable lead-lag motion must then all be due to nonlinear effects. Furthermore, the flapping motion for $\mu = 1.4$ at the higher azimuth angles appears to have been excited by the persisting lead-lag motion, resulting in an apparent damping of the motion which is much less than would otherwise be the case.

Comparison with Experimental Data

Gates and DuWaldt[1] report the results of an experimental investigation of the flutter of a model helicopter rotor in forward flight. The model rotor has a single blade with a radius of four feet and with a flapping hinge through the axis of rotation. The blade had a constant chord of 3.5 in. and a root cutout of 6 in. The blade was relatively stiff in torsion, but the control system was made flexible, so the primary contributions to blade motions derived from rigid-body pitch, flapping motions and deflections in the first flapwise bending mode. The main parameters of the study were advance ratio, control system stiffness and chordwise mass center location.

The case selected for comparison had a ratio of first flapwise bending frequency $\bar{\omega}_{\phi_1}$ to control frequency $\bar{\omega}_{\theta_0}$ (nonrotating) of 1.31 and a chordwise mass center location of 42.5% of chord aft of the leading edge, giving a value of 0.139 for equivalent mass center location as defined by Gates and DuWaldt.[1] This case was chosen because the experimental data as plotted in Fig. 8e of Ref. 1 indicated there should be a relatively large change in stability with advance ratio. Natural frequencies and mode shapes for the first three coupled modes of the blade were calculated for a value of the ratio $\bar{\omega}_{\theta_0}/\Omega = 4.37$, and Fourier coefficients of the coefficients of the equations of motion were calculated for values of advance ratio μ of 0.0, 0.09, 0.175, 0.24, and 0.30. The Fourier coefficients for each value of μ were supplied to the main computer program and the system characteristic values computed. These points were selected for calculation so that some would lie in a stable and some in an unstable region, as determined experimentally.

Subsequent to these calculations, it was determined through personal communications that the data of Fig. 8e of Ref. 1 were mislabelled. The symbols for values of $\bar{\omega}_{\phi_1}/\bar{\omega}_{\theta_0}$ of 1.31 and 0.63 were reversed. The experimentally determined flutter boundary actually corresponding to the calculations performed, consisting of a plot of $\bar{\omega}_{\theta_0}/\Omega$ vs μ, taken from Fig. 8e of Ref. 1, is reproduced in Fig. 4. The points at which characteristic values were calculated are indicated by asterisks on the line drawn at $\bar{\omega}_{\theta_0}/\Omega = 4.37$. Unfortunately, as can be seen in Fig. 4, all the calculated points lie in a stable region with respect to the correct data points,

Table 2 Characteristic values for comparison with experimental data

$\mu = 0.0$		$\mu = 0.09$	
λ_R	λ_I	λ_R	λ_I
−0.06273	0.02028	−0.06271	0.02800
−0.06273	−0.02028	−0.06271	−0.02800
−0.33052	0.20163	−0.33055	0.20185
−0.33052	−0.20163	−0.33055	−0.20185
−0.47015	0.08874	−0.47016	0.08954
−0.47015	−0.08874	−0.47016	−0.08954
$\mu = 0.175$		$\mu = 0.24$	
λ_R	λ_I	λ_R	λ_I
−0.06266	0.02871	−0.06268	0.02957
−0.06266	−0.02871	−0.06268	−0.02957
−0.33068	0.20267	−0.33120	0.20366
−0.33068	−0.20267	−0.33120	−0.20366
−0.47012	0.09152	−0.47010	0.09342
−0.47012	−0.09152	−0.47010	−0.09342

$\mu = 0.30$	
λ_R	λ_I
−0.06283	0.03060
−0.06283	−0.03060
−0.33227	0.20489
−0.33227	−0.20489
−0.47013	0.09510
−0.47013	−0.09510

and time limitations prevented carrying out more extensive calculations. However, some information can still be derived from the calculations which were performed. The characteristic values obtained are listed in Table 2.

The results of the calculations indicate, first, that the stability of the rotor for the control stiffness selected is essentially independent of advance ratio over the range of μ considered. The relative insensitivity to advance ratio changes is certainly in agreement with the plot of Fig. 4. Secondly, it should be noted that the rotor is predicted to be very nearly unstable. The value for λ_R of -0.063 indicates that about ten rotor revolutions are required to damp the motion by a factor e^{-2}. The limited calculations which were performed, then, are at least in qualitative agreement with the experimental results. A definitive quantitative comparison must await more extensive calculations.

Concluding Remarks

A method has been developed for analyzing the stability of dynamic systems with periodically varying parameters. The method employs formulations for calculating the characteristic values of the perturbation equations of motion, the latter being a coupled set of second-order, linear differential equations with periodic coefficients. The characteristic values, which are the zeros of an infinite determinant, are calculated from an equivalent analytic form for the infinite determinant consisting of a finite sum of hyperbolic functions.

A digital computer program was prepared which implements the method for systems with one, two or three degrees of freedom. Calculations were carried out for comparison with results of a direct time integration on a digital computer of the equations for a rigid rotor blade with flapping and lead-lag hinges. The results were in qualitative agreement. Quantitative differences were attributable to the nonlinear effects which were included in the direct time integration. Also, limited calculations were performed for comparison with experimentally derived flutter boundaries for a model rotor blade with three degrees of freedom. The calculations indicate that the rotor is only marginally stable at the control stiffness selected and that the stability is relatively insensitive to advance ratio, in agreement with the experimental results.

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